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Application of Jumarie's Fractional Derivative to Degassing a Thin Plate

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In this work, we have dealt with a problem encountered in transport phenomena. The equations describing such phenomenon contain fractional derivatives. We use the modified Jumarie's definition of such a derivative to solve the transport equation. In particular, we have treated the space–time fractional diffusion equation (of Fick's law) regarding the process of degassing a thin plate in vacuum.

У цій роботі ми розглядаємо проблему, що виникає в явищах перенесення. Рівняння, що описують таке явище, містять дробові похідні. Ми використовуємо модифіковане визначення за Джумарі такої похідної для розв'язання рівняння перенесення. Зокрема, ми розглядаємо просторово-часове дробове дифузійне рівняння (за Фіковим законом) стосовно процесу дегазації тонкої пластини у вакуумі.

Key words: fractional Jumarie's derivative, Fick's law, Mittag-Leffler functions, Laplace transform.

Ключові слова: дробова похідна за Джумарі, Фіків закон, функції Міттаґ-Леффлера, Ляплясів перетвір.

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1. INTRODUCTION

In recent decades, considerable attention has been paid to the fractional derivative by the application of this concept in different areas of physics as quantum physics [1–6], quantum electronics, nanoelectronics, transport in nanostructures [7–10], and in general engineering as: continuum mechanics, viscoelastic and viscoplastic flow, electrical circuits, control theory, image processing, viscoelasticity, biology and hydrodynamics [11–15]. Historically, the fractional calculus has been developed by Riemann and Liouville. Not only does the latter defines a derivative and antiderivative (integral) for an integer order (as usual derivative of order one, two, *etc.*), but to give meaning to the derivative with non-integer order. The fractional calculus was developed recently in Refs. [16–20] and applied successfully for modelling some physical processes [21–29]. One important of such process is the diffusion phenomenon. The associated fractional diffusion equation arises quite naturally in continuous-time random walks. The fractional derivatives may be introduced by different definitions. For example, Jumarie [30] has considered the Riemann–Liouville definition and modified it agreeing with the fractional difference definition and fully consistent with the fractional-difference definition and avoiding any reference to the derivative of order greater than the considered ones.

The purpose of this paper is to study the problems of transport equation as diffusion equation (Fick's equation) in ordinary space. Section 2 starts with defining fractional Jumarie's derivative [30]. We show that this definition is agreed with the standard derivative; as it happens, the fractional derivative of a constant is well zero. Furthermore, when we perform the limit $\alpha = 1$, the standard case is recovered. In Section 3, we treated the Fick's equation, when the derivatives (in time and space) are fractional ($0 \le \alpha \le 1$ and $0 \le \beta \le 2$). The solution of this equation is presented explicitly for a particular case. We end this work with a conclusion in Sec. 4.

2. JUMARIE'S DERIVATIVE

The analytical solutions of the fractional differential equation are emerging branch of applied science also in basic science such as applied mathematics, physics, mathematical biology, and engineering. There are many types of fractional integral and differential operators with the Riemann–Liouville (R–L) definition. Other useful definition includes the Caputo definition of fractional derivative (1967). Riemann–Liouville definition of the fractional derivative of a constant is non-zero that creates a difficulty to relate between the basic calculi. To overcome this difficulty, Jumarie modified the definition of fractional derivative of Riemann–Liouville type as follows [30]:

$$
{}_aD_t^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(-\alpha)}\displaystyle\int_0^t (t-\xi)^{-\alpha-1} f(\xi)d\xi, \ \alpha < 0, \\ \dfrac{1}{\Gamma(1-\alpha)}\left(\dfrac{d}{dt}\right)\displaystyle\int_0^t (t-\xi)^{-\alpha}\left[f(\xi)-f(0)\right]d\xi, \quad 0 \le \alpha < 1, \\ \left[f^{(\alpha-n)}(t)\right]^n, \ 0 \le \alpha < 1, \ n \ge 1. \end{cases} \quad (1)
$$

With this new formulation, we obtain the derivative of a constant as zero. For $\alpha = 1$, we can check that the second definition recovers the standard first derivative of $f(t)$; to be convinced, it easy to see this statement by performing the Laplace transform to both side of the second definition. We must stop here to make an important remark about the second definition of Jumarie ($0 \le \alpha < 1$):

$$
{}_aD_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{d}{dt}\right)\int_0^t (t-\xi)^{-\alpha}\left[f(\xi)-f(0)\right]d\xi = \lambda\frac{d}{dt}g(t)\,; \quad (2)
$$

we can write it as follows:

$$
\int_0^t (t-\xi)^{-\alpha}\left[f(\xi)-f(0)\right]d\xi = \lambda\Gamma(1-\alpha)g(t)\,. \quad (3)
$$

Recently in Ref. [26], it has developed analytical method for solution of linear fractional differential equations with Jumarie's type derivative in terms of Mittag-Leffler functions and found that the solution of the fractional differential equation ${}^jD_t^\alpha y = ay$ is $y = E_\alpha(ax^\alpha)$. This new finding has been extended in Ref. [31] to get analytical solution of system of linear fractional differential equations. The main aim of this work is to investigate the possibility of applying these new analytical-solution methods for treatment the space–time fractional diffusion equation with Jumarie's type derivative for degassing a thin plate in vacuum.

3. APPLICATIONS

In this section, we will deal with the Fick's equation describing the diffusion of impurities in a material during the process degassing a thin plate under vacuum, which allows industry of a high-strength component to reduce the impurity (like hydrogen) content within the material by being removed in gas form [32, 33]. The impurity transport in the material during degassing can be modelled by the space–time fractional diffusion equation, which is referred to as

$$D_t^{\alpha} c(x,t) = \mu D_x^{\beta} c(x,t)\,, \tag{4}$$

where μ is the diffusion coefficient of impurity through the material; $0 < \alpha \le 1$ and $0 < \beta \le 2$.

Consider the case of degassing a thin plate of thickness L in vacuum, whose surfaces, $x = 0$, $x = L$, are maintained at zero concentration of impurity. The initial concentration of impurity is a given function $c_0(x)$ defined at all points x on the plate. Therefore, the initial and boundary conditions can be written as the following equations:

$$c(x,0) = c_0(x),\ 0 < x < L;\ c(0,t) = c(L,t) = 0\,. \tag{5}$$

As μ is time-independent, we use the separation of variables technique:

$$c(x,t) = c_1(t) c_2(x)\,, \tag{6}$$

to get the following two equations to solve:

$${}^{j}D_t^{\alpha} c_1(t) = -k^2 \mu c_1(t)\,, \tag{7}$$

which is a time-dependent equation, and

$${}^{j}D_x^{\beta} c_2(x) = -k^2 c_2(x)\,, \tag{8}$$

which is a space-dependent equation, and k is a positive real constant.

3.1. Time-Dependent Equation

Taking into account the Jumarie's definition (1), we can transform Eq. (7) to the following integral equation:

$$-k^2 \mu c_1(t) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{d}{dt} \right) \int_0^t (t-\xi)^{-\alpha} \left[c_1(\xi) - c_1(0) \right] d\xi\,. \tag{9}$$

By taking Laplace transform in both side, assumed that $C_1(p) = L(c_1(t))$, it is easy to find

$$C_1(p) = \frac{c_1(0) p^{\alpha-1}}{k^2 \mu + p^{\alpha}}\,. \tag{10}$$

Then, by taking the inverse of Laplace transform, we find the solution of the time-dependent Eq. (7) as follows:

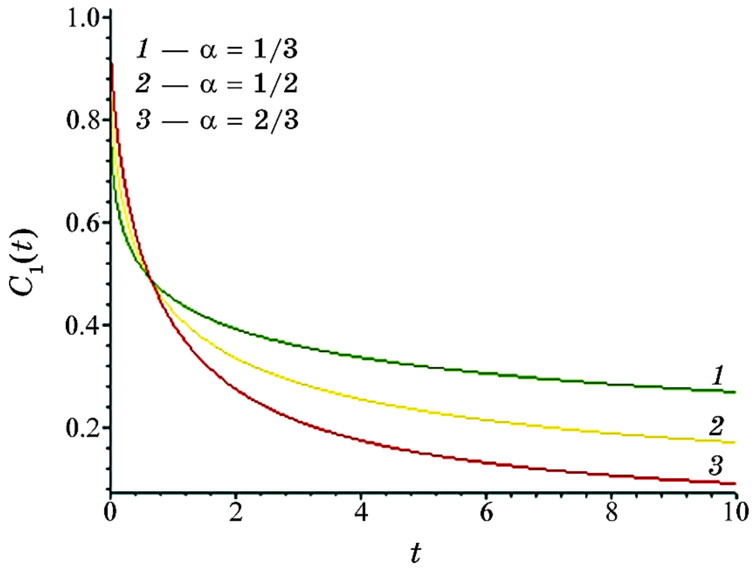

Fig. Time-dependent part of the concentration for different values of α.

$$c_1(t) = \oint_\gamma C_1(p)\exp(pt)dp = c_1(0)\oint_\gamma \frac{p^{\alpha-1}}{k^2\mu + p^\alpha}\exp(pt)dp \equiv E_\alpha(-\mu k^2 t^\alpha) , \tag{11}$$

which is shown in Fig. for different values of α.

For example, in case $\alpha = 1/2$, we get:

$$c_1(t) = E_{1/2}(-\mu k^2\sqrt{t}) = \exp(\mu^2 k^4 t)\mathrm{erfc}(\mu k^2\sqrt{t}) . \tag{12}$$

It is clear that, when we put $\alpha = 1$ in Eq. (11), we retrieve the standard case.

3.2. Space-Dependent Equation

To solve Eq. (8), we put $\beta = 2\gamma$, where $0 < \gamma \le 1$; therefore, Eq. (8) can be written as

$$^{jum}D_x^{2\gamma}c_2(x) = -k^2 c_2(x) , \tag{13}$$

From Ref. [26], the solution of the last equation is

$$c_2(x) = A\cos_\gamma(kx^\gamma) + B\sin_\gamma(kx^\gamma) , \tag{14}$$

where

$$\cos_\gamma(kx^\gamma) = \sum_{n=0}^{\infty}(-1)^n (kx^\gamma)^{2n}/\Gamma(2n\alpha+1)\,, \tag{15}$$

$$\sin_\gamma(kx^\gamma) = \sum_{n=0}^{\infty}(-1)^n (kx^\gamma)^{2n+1}/\Gamma(2n\alpha+\alpha+1) \tag{16}$$

are the fractional cosine and sine. Therefore, the solution for the space–time concentration, governed by Eq. (4), can be written as

$$c(x,t) = E_\alpha(-k^2\mu t^\alpha)\left[A\cos_\gamma(kx^\gamma) + B\sin_\gamma(kx^\gamma)\right], \tag{17}$$

where A and B are constants to be determined by using the boundary conditions (5). Using the boundary conditions, we observe that

$$A = 0,\ kL^\gamma = (nM)^\gamma\,, \tag{18}$$

where M is the period [26]. Then, we get

$$c(x,t) = B_\bullet E_\alpha\left(-\left(\frac{(nM)^\gamma}{L^\gamma}\right)^2 \mu t^\alpha\right)\sin_\gamma\left((nM)^\gamma \frac{x^\gamma}{L^\gamma}\right). \tag{19}$$

The general solution is a linear combination

$$c(x,t) = \sum_{n=0}^{\infty} B_n E_\alpha\left(-\mu\left(\frac{(nM)^\gamma}{L^\gamma}\right)^2 t^\alpha\right)\sin_\gamma\left((nM)^\gamma \frac{x^\gamma}{L^\gamma}\right). \tag{20}$$

Using the initial conditions (5), we have

$$c(x,0) = c_0(x) = \sum_{n=0}^{\infty} B_n \sin_\gamma\left((nM)^\gamma \frac{x^\gamma}{L^\gamma}\right). \tag{21}$$

To determine the coefficients B_n, we multiply the both sides of the last equation by $\sin_\gamma\left((mM)^\gamma \frac{x^\gamma}{L^\gamma}\right)$, where m is an integer, and integrate both sides of the resulting equation from a zero to L, and we get the following result:

$$\int_0^L c_0(x)\sin_\gamma\left((mM)^\gamma \frac{x^\gamma}{L^\gamma}\right)dx = \sum_{n=0}^{\infty} B_n \int_0^L \sin_\gamma\left((mM)^\gamma \frac{x^\gamma}{L^\gamma}\right)\sin_\gamma\left((nM)^\gamma \frac{x^\gamma}{L^\gamma}\right)dx \equiv \equiv B_m, \tag{22}$$

or equivalently

$$B_n = \int_0^L c_0(y)\sin_\gamma\left((nM)^\gamma \frac{y^\gamma}{L^\gamma}\right)dy \tag{23}$$

that gives the global solution of the diffusion Eq. (4) as

$$c(x,t) =$$
$$= \sum_{n=0}^{\infty} E_\alpha\left(-\mu\left(\frac{(nM)^\gamma}{L^\gamma}\right)^2 t^\alpha\right)\sin_\gamma\left((nM)^\gamma \frac{x^\gamma}{L^\gamma}\right)\int_0^L c_0(y)\sin_\gamma\left((nM)^\gamma \frac{y^\gamma}{L^\gamma}\right)dy. \tag{24}$$

As initial function, we take the initial spatial concentration as

$$c_0(y) = c_0 y(L-y)\,. \tag{25}$$

Then, the coefficient B_n (23) becomes as follows:

$$B_n = c_0\int_0^L y(L-y)\sin_\gamma\left((nM)^\gamma \frac{y^\gamma}{L^\gamma}\right)dy\,. \tag{26}$$

By replacing the fractional sine and making the change $y = Lh$, we find

$$B_n = c_0 L^3\int_0^1 h(1-h)\sum_{k=0}^{\infty}(-1)^k \frac{\left((nM)^\gamma h^\gamma\right)^{2k+1}}{\Gamma(2k\alpha+\alpha+1)}dh \tag{27}$$

or equivalently

$$B_n = c_0 L^3\sum_{k=0}^{\infty}(-1)^k \frac{(2k+1)\gamma+1}{\Gamma\left((2k+1)\gamma+4\right)}\left((nM)^\gamma\right)^{2k+1}. \tag{28}$$

The last expression can be seen to be replaced by

$$B_n = c_0 L^3 (nM)^\gamma\left[\sum_{k=0}^{\infty}\frac{(-\lambda_n)^k}{\Gamma\left((2k+1)\gamma+3\right)} - 2\sum_{k=0}^{\infty}\frac{(-\lambda_n)^k}{\Gamma\left((2k+1)\gamma+4\right)}\right] \tag{29}$$

or equivalently

$$B_n = c_0 L^3 (nM)^\gamma\left[E_{2\gamma,\gamma+3}(-\lambda_n) - 2E_{2\gamma,\gamma+4}(-\lambda_n)\right], \tag{30}$$

such that $E_{\mu,\nu}(x)$ is the Mittag-Leffler function of the second kind, and

$$\lambda_n = (nM)^{2\gamma}\,, \tag{31}$$

leading to the following final closed result ($\gamma = \beta/2$):

$$c(x,t)=\sum_{n=0}^{\infty}E_{\alpha}\left(-\mu\left(\frac{\sqrt{\lambda_n}}{L^{\gamma}}\right)^2 t^{\alpha}\right)B_n\sin_{\gamma}\left(\sqrt{\lambda_n}\frac{x^{\gamma}}{L^{\gamma}}\right), \quad (32)$$

$$c_0L^3\sum_{n=0}^{\infty}E_{\alpha}\left(-\mu\left(\frac{\sqrt{\lambda_n}}{L^{\gamma}}\right)^2 t^{\alpha}\right)\sqrt{\lambda_n}E_{2\gamma,\gamma+3}(-\lambda_n)\sin_{\gamma}\left(\sqrt{\lambda_n}\frac{x^{\gamma}}{L^{\gamma}}\right)=$$

$$=-2c_0L^3\sum_{n=0}^{\infty}E_{\alpha}\left(-\mu\left(\frac{\sqrt{\lambda_n}}{L^{\gamma}}\right)^2 t^{\alpha}\right)\sqrt{\lambda_n}E_{2\gamma,\gamma+4}(-\lambda_n)\sin_{\gamma}\left(\sqrt{\lambda_n}\frac{x^{\gamma}}{L^{\gamma}}\right). \quad (33)$$

4. CONCLUSION

In this work, we have treated some problems, which encountered in transport phenomena. The equations describing these phenomena have fractional derivatives. At first, we have presented the modified Jumarie's definition of such derivatives. After what, we have, as application, solved the fractional diffusion equation (Fick's law) that presents a partial fractional differential on the time t and a partial fractional differential on the spatial co-ordinate x. The order of the fractionality in the time is α $(0 \le \alpha < 1)$ and in the space is β $(0 \le \beta < 2)$. The solution of the time fractional equation is expressed in term of Mittag-Leffler function of the first kind, whereas the solution of the spatial fractional equation is expressed in term of Mittag-Leffler function of the second kind (see Eq. (33)). This treatment can be useful to describe the transport phenomena in nanomaterials and nanostructures.

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